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Annealed and Quenched Inhomogeneous Cellular Automata (INCA)

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A probabilistic one-dimensional cellular automaton model by Domany and Kinzel is mapped into an inhomogeneous cellular automaton with the Boolean functions XOR and AND as transition rules. Wolfram's classification is recovered by varying the frequency of these two simple rules and by quenching or annealing the inhomogeneity. In particular, "class 4" is related to critical behavior in directed percolation. Also, the critical slowing down of second-order phase transitions is related to a stochastic version of the classical "halting problem" of computation theory.

KEY WORDS: Directed percolation; inhomogeneous cellular automata; cellular automata classification; phase transitions; complexity.

1. INTRODUCTION

Invented in 1948 by John von Neumann and Stanislaw Ulam,[14,2] cellular automata consist of regular arrays of cells with a state (0 or 1, in this paper) at each cell. With an evolution law in discrete time, a cellular automaton is a fully discrete dynamical system. In the simplest one-dimensional case, the value a_i^{t+1} that a cell i takes at time $t+1$ is determined by the values a_{i-1}^t and a_{i+1}^t assumed at time t by the adjacent neighboring cells. (For an introduction, see Ref. 8, and for reviews, see Ref. 4.) Wolfram[15] has proposed a classification of one-dimensional cellular automata according to the possible evolutions of a chain initially filled at random with 0's and 1's. The classification involves four classes:

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1. The chain becomes homogeneous (say, all 0's).
2. Appearence of simple localized time-periodic structures.
3. Evolution leads to chaotic behavior.
4. Evolution leads to complex localized structures.

An exhaustive study of the simplest one-dimensional cellular automata (two states per cell, dependence on nearest neighbors only) shows they do not exhibit class 4 behavior (except perhaps by rule 193). This behavior is displayed only by automata that involve more states per cell or a wider neighborhood. It should be noted, however, that this conclusion affects deterministic cellular automata only. But Wolfram's classification, proposed for deterministic cellular automata, can also be used as a phenomenological characterization of probabilistic cellular automata, in which the transition rule is stochastic. Probabilistic cellular automata have recently been the object of intense investigations.[1,5,6,13]

2. DIRECTED PERCOLATION

Domany and Kinzel (see Refs. 3 and 10 for more details) have investigated in detail the connection between one-dimensional stochastic cellular automata and two-dimensional directed percolation. Indeed, the space-time graph of one-dimensional cellular automata (where successive chain configurations are plotted below each other) can be seen as a static two-dimensional percolation pattern. The arrow of time corresponds to the directedness of the percolation problem: row t corresponds to time step t. Domany and Kinzel have studied in particular a simple model in which the considered cell i takes the value 1 according to a conditional probability $P(a_i^{t+1}=1 \mid a_{i-1}^t, a_{i+1}^t)$, which depends on the present values (a_{i-1}^t, a_{i+1}^t) of the adjacent neighbors. Since obviously

$$P(a_i^{t+1}=0 \mid a_{i-1}^t, a_{i+1}^t) = 1 - P(a_i^{t+1}=1 \mid a_{i-1}^t, a_{i+1}^t)$$

the model is entirely described by the definition of the four P's for the four possible values of the neighbors (a_{i-1}^t, a_{i+1}^t). Domany and Kinzel have reduced the number of parameters from four to two by imposing a left–right symmetry and by requiring that the quiescent configuration ($a_i=0$ for all i) be absorbing—the system has no source. The dynamics is now entirely defined by the values of two parameters p_1 and p_2:

$$P(1 \mid 1, 0) = P(1 \mid 0, 1) = p_1$$

$$P(1 \mid 1, 1) = p_2$$

$$P(1 \mid 0, 0) = p_0 = 0$$

The actual mapping between this cellular automaton and an instance of the directed percolation problem is achieved by considering the (two-dimensional) graph of the evolution of the (one-dimensional) cellular automaton and by the assignment[3,10]

$$p_1 = p_s p_b$$

$$p_2 = p_s p_b(2 - p_b)$$

where p_s and p_b are the site and bond percolation probabilities, respectively.

Domany and Kinzel have observed a percolation transition between two phases (dry and wet). The phase is dry if the quiescent configuration (all zeros) is reached at some t for almost all initial chains. Otherwise the phase is wet, i.e., some initial 1's manage to percolate and "wet" chaotically the chain at time t. In Fig. 1 (adapted from Refs. 3 and 10) the wet phase is at the right of the thin line in the (p_1, p_2) space. Notice that in terms of Wolfram's classification, the dry and wet phases correspond to class 1 and 3 cellular automata, respectively.[10]

In this paper, we further simplify Domany and Kinzel's model by focusing on the line $p_2 = 1 - p_1$ (the thick line in Fig. 1). Reducing the number of parameters to one yields an advantage: it endows the

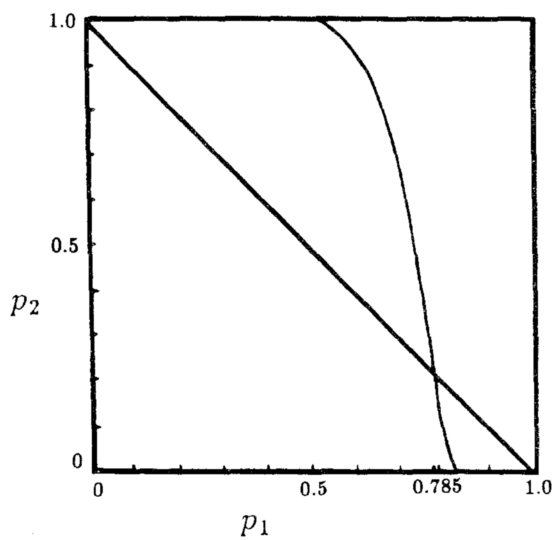

Fig. 1. Phase diagram of the two-parameter model by Domany and Kinzel.[3,10] The wet phase is at the right of the thin line. The present paper focuses on the line $p_2 = 1 - p_1$. This line can be interpreted in terms of the XOR and AND Boolean functions.

probabilities p_1 and p_2 with a direct meaning in terms of the familiar Boolean functions AND and Exclusive OR (XOR). A cell will define its future state a_i^{t+1} by taking the XOR (with probability p^{XOR}) or the AND (with probability p^{AND}) of the present values a_{i-1}^t and a_{i+1}^t of its neighbors. This interpretation applies to Domany and Kinzel's model (on the line $p_2 = 1 - p_1$), by equating p^{XOR} with p_1 and $p^{\mathrm{AND}} = 1 - p^{\mathrm{XOR}}$ with p_2. [For example, $P(0|11)$ corresponds in Domany and Kinzel's model to $1 - p_2$, i.e., to p_1 with the constraint $p_1 + p_2 = 1$, and to p^{XOR} in the present discussion.] The functions AND and XOR return $a_i^{t+1} = 0$ when the inputs a_{i-1}^t and a_{i+1}^t are both zero, and thus agree with Domany and Kinzel's assignment $p_0 = 0$.

The line $p_2 = 1 - p_1$ crosses Domany and Kinzel's transition line at a critical value $p_c = 0.785$ of $p_1 = p^{\mathrm{XOR}}$. We of course recover Domany and Kinzel's findings in our measurements for $p^{\mathrm{XOR}} = 0.77$ ($< p_c$) (Fig. 2) and for $p^{\mathrm{XOR}} = 0.81$ ($> p_c$) (Fig. 3). These figures show the evolution of an 896-cell-long automaton with circular boundary conditions. At $t = 0$ (top row), we distribute zeros and ones at random on the central interval of length 300, and out of this interval zeros only. Since a_i^{t+1} depends on the present values a_{i-1}^t and a_{i+1}^t of the neighbors and *not* on the present value a_i^t of the

Fig. 2. Evolution with $p^{\mathrm{XOR}} < p_c$ (annealed inhomogeneity). Each cell updates its state following the XOR rule with a probability $p^{\mathrm{XOR}} = 0.77$. The system dries up after 460 steps (class 1 behavior).

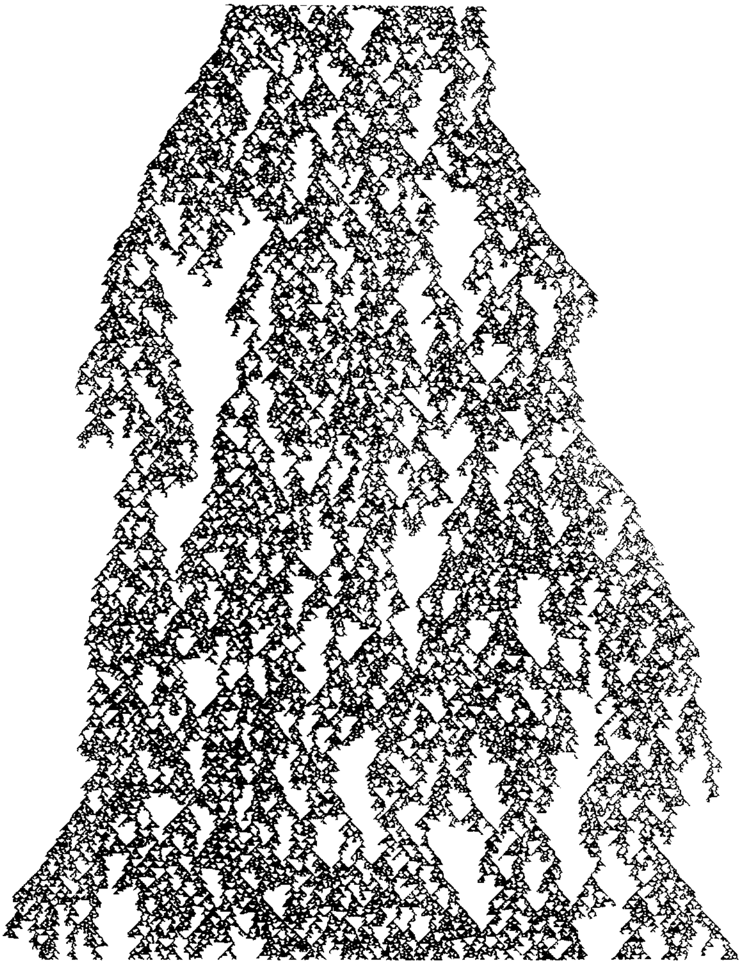

Fig. 3. Evolution with $p^{XOR} > p_c$ (annealed inhomogeneity). Each cell updates its state following the XOR rule with a probability $p^{XOR} = 0.81$. The system forms a chaotic pattern that extends in width as well as in the time direction (class 3 behavior).

considered cell itself, the automaton splits into two *noninteracting* systems that are interweaved in a checkerboard pattern in the (i, t) lattice. To single out one of these systems, we assign at $t = 0$ ones at random on even sites only. We stop the experiment at the first occurrence of one of the following:

1. The system dries out.
2. The system reaches 10,000 generations.
3. The pattern's width has doubled with respect to the original value (300).

We observe that the system dries up below the critical point ($p^{XOR} < p_c$), i.e., when the cells update their states by following the XOR rule with a frequency lower than 78.5%. We have in this case an instance of a class 1 automaton (Fig. 2). In contrast, above the critical point, the 1's propagate in the time direction, and the corresponding correlation length $\xi_{\parallel}$ diverges. What is more, the nonzero region becomes wider and wider, and the perpendicular correlation length $\xi_{\perp}$ diverges as well (Fig. 3). The pattern of 1's in the growing fan shows no order; they are distributed in a chaotic way. We thus have a class 3 automaton when the cells update their states by following the XOR rule with a frequency larger than 78.5%.

3. CLASS 4 BEHAVIOR AND CRITICAL SLOWING DOWN

At precisely $p^{XOR} = p_c$, we ran 128 simulations with 10,000 evolution steps from various initial conditions. We observe that 29% of the runs dry up (as in Fig. 2), 53% show chaotic growth (as in Fig. 3), and 18% display a peculiar behavior: the region occupied by the cells in state 1 extends in the time direction but remains almost constant in width. The nonzero band shows oscillations of small amplitude around the original width. Figure 4 shows the first 1000 steps of one of these simulations. This behavior occurs in an immediate neighborhood of p_c only.

The patterns are complex, and *localized*; they are characteristic of class 4 behavior. If our phenomenon is a second-order transition, then it exhibits critical slowing down, producing transients with unbounded lifetimes. The fate of an initial line evolving under the critical value $p^{XOR} = p_c$ is essentialy unpredictable. This is very similar to the halting problem in the theory of computation, according to which no algorithm can decide whether a given pair of program and initial data will halt or not.[12] However, it is an analogy rather than an identity, since, strictly speaking, the halting problem is defined for deterministic computations only. Wolfram's type 4 behavior in our system seems to be related to critical behavior in directed percolation; it will require a great deal more work to explore this connection. We can already observe, however, that classes 1, 3, and 4 can be obtained from one another by varying a single parameter (p^{XOR}) in a model that involves two very simple rules. Furthermore, the data suggest that class 4 corresponds to a set of measure zero in this parameter space.

4. QUENCHED INHOMOGENEOUS AUTOMATA

By mixing with various probabilities two very simple rules, we recover three out of four of Wolfram's behaviors. The question remains if there is a way to combine the XOR and AND rules and provide a model for class 2

Fig. 4. The first 1000 steps of an evolution at the critical point $p^{\mathrm{XOR}} = p_c$ (annealed inhomogeneity). Each cell updates its state following the XOR rule with the critical probability $p^{\mathrm{XOR}} = 0.785$. The patterns are localized and complex (class 4 behavior).

behavior. It turns out that the type 2 behavior can be obtained by simply *quenching* the inhomogeneity of the transition rule. In other words, we distribute the rules XOR and AND once and for all at $t = 0$, and we quench or freeze this distribution for all later times. In this limit, the updating is disorderly in space, but it becomes purely deterministic. In contrast, when the distribution of rules is refreshed at each time step, as in the examples discussed above, the inhomogeneity actually disappears: it is wholly carried by the varying occupations of the neighborhood (a_{i-1}^t, a_{i+1}^t). Since the inhomogeneity of the system has "melted" in the stochasticity of the dynamics, we call such systems *annealed* inhomogeneous cellular automata,

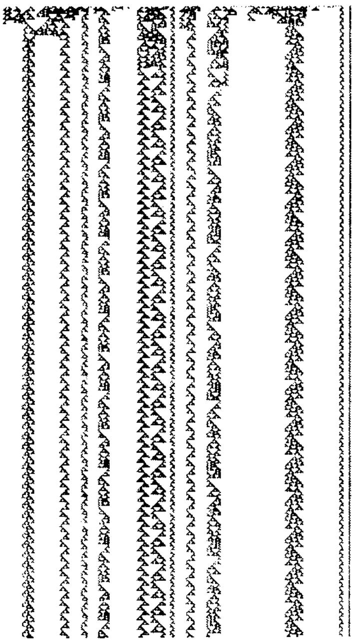

Fig. 5. Quenched inhomogeneity. Evolution when 80% of the cells follow the XOR rule at all times. The other cells update their state following the AND rule. The pattern becomes simple, localized, and time-periodic (class 2 behavior).

to contrast them with the quenched systems. As Fig. 5 shows, the patterns generated with quenched automata are typical of class 2: after a short transient, they become localized, but periodic and fairly simple.

In two dimensions, the phenomena of the quenched distributions of XOR and AND rule becomes much richer.[7] The second dimension allows for various feedback loops, and we obtain clusters of 1's with enormous cycle times. The percolation structure of such clusters accounts for much of Kauffman's findings on Boolean networks. The motivation of using in two dimensions the AND and XOR rules stems from the fact that these rules are respectively *canalizing* (of *forcing*)[3] and *noncanalizing* (or *nonforcing*), according to an important characterization due to Kauffman.[9]

[3] In a canalizing rule, the value of at least one neighbor suffices to guarantee the next state of the cell, regardless of the other neighbor's values. In our model the XOR sites are nonforcing, as the knowledge of the output is not "forced" by one of the inputs, whereas the next value at an AND site is forced by a 0 at any of the neighbors.

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